Angular Momentum in General Relativity

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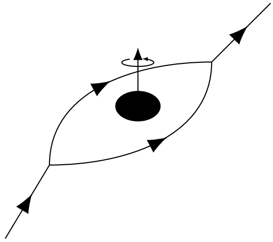
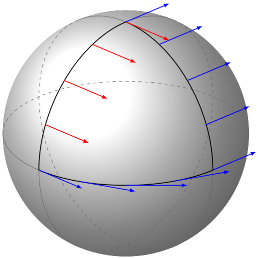




Outline

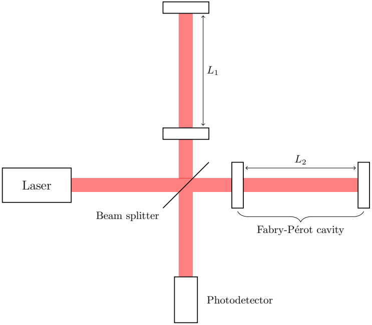
I. A Notion of Angular Momentum in Black Hole Spacetimes

II. Angular Momentum and Gravitational Wave Memory

III. ~~Angular Momentum in Einstein-Maxwell Theory~~

Context: Gravitational wave astronomy

- First detection by LIGO: binary black hole merger (GW150914)
- *Undetectable* by conventional astronomy!



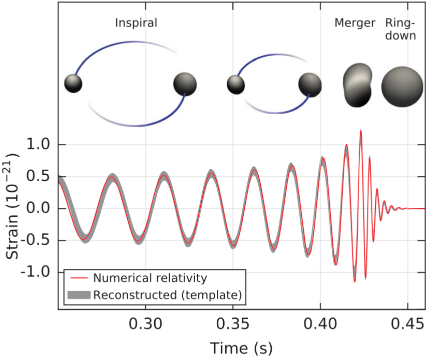


Image credit: Phys. Rev. Lett. **116**, 061102 (2016)

Highlights of detections

- Black hole mergers: probes population/formation
- Neutron star mergers:
  - Nuclear matter EoS
  - EM counterparts
- Independent probe of cosmic expansion (with localization)
- General constraints on modified gravity

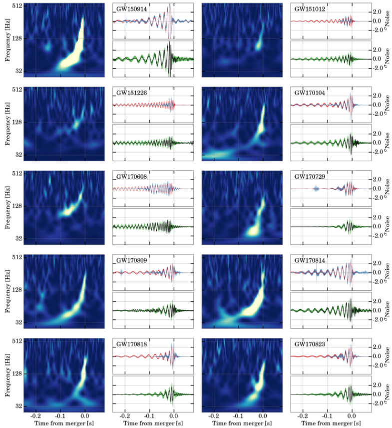
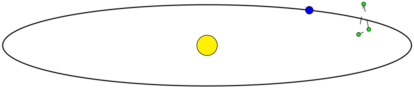


Image credit: Phys. Rev. X **9**, 031040 (2019)

Future space-based detector



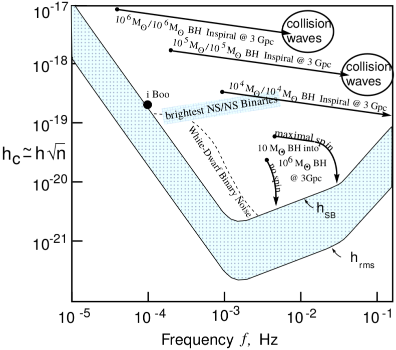


Image credit: arXiv:gr-qc/9506086

[Thorne, 1995]

- LISA Mission: $\sim$ 2034
- Lower $f \implies$ higher-mass mergers
- LISA Pathfinder: exceeded design sensitivity (!)

Outline

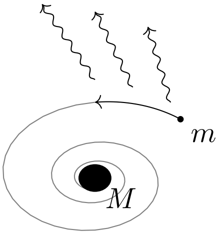
I. A Notion of Angular Momentum in Black Hole Spacetimes

II. Angular Momentum and Gravitational Wave Memory

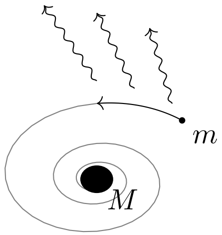
III. ~~Angular Momentum in Einstein-Maxwell Theory~~

Background: Extreme mass-ratio inspirals (EMRIs)

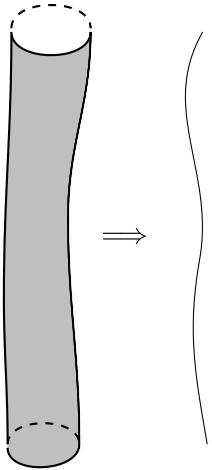
- Astrophysics: stellar-mass objects orbiting supermassive black holes
- Mass ratio: $m/M \lesssim 10^{-5}$
- Needs LISA: $M \gtrsim 10^6 M_\odot$
- Signal maps out geometry of black hole spacetime
- Tests "no-hair theorem": black holes only characterized by mass & spin



Self-force



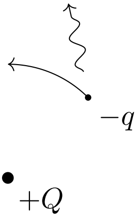
- $m/M \ll 1$: suggests perturbative expansion for EMRIs
- 0^{th} order: emission from stable orbit
- Higher orders: gravitational “self-field” influences orbit
- Corrections to geodesic motion in point particle limit



Flux-balance laws (classical electromagnetism)

- Orbits characterized by
  $E = p_a t^a$, $L_z = p_a \phi^a$
  (t, ϕ symmetries)
- Averaged loss of (e.g.) E:

$$\left\langle \frac{\mathrm{d}E}{\mathrm{d}\tau} \right\rangle = \underbrace{\left\langle \frac{\mathrm{d}E_\infty}{\mathrm{d}\tau} \right\rangle}_{\text{lost to infinity}}$$



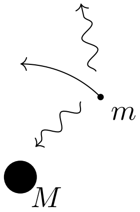
- Conserved currents: "E, L_z of matter & fields"

$$j_t^a \equiv T^a{}_b t^b, \qquad j_\phi^a \equiv T^a{}_b \phi^b$$

Flux-balance laws (non-spinning black holes)

- Orbits characterized by
  $E = p_a t^a$, $L_z = p_a \phi^a$
  (t, ϕ symmetries)
- Averaged loss of (e.g.) E:

$$\left\langle \frac{\mathrm{d}E}{\mathrm{d}\tau} \right\rangle = \underbrace{\left\langle \frac{\mathrm{d}E_\infty}{\mathrm{d}\tau} \right\rangle}_{\text{lost to infinity}} + \overbrace{\left\langle \frac{\mathrm{d}E_{\text{hor.}}}{\mathrm{d}\tau} \right\rangle}^{\text{lost to black hole}}$$

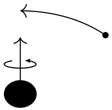


- Conserved currents: "E, L_z of matter & fields"

$$j_t^a \equiv T^a{}_b t^b, \qquad j_\phi^a \equiv T^a{}_b \phi^b$$

Spinning black holes: the Carter constant

- Not spherically symmetric
  $\implies$ no "total angular momentum" $\vec{L}$
  $\implies$ non-planar motion
- Additional constant of motion required:
  Carter constant K [Carter, 1968]
- Non-spinning: $K = L^2$
- $\left\langle \dfrac{\mathrm{d}K}{\mathrm{d}\tau} \right\rangle$ required to determine inspiral
- *No* conserved K constructed from T^{ab}
  [G & Flanagan, 2015]

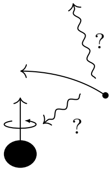


Spinning black holes: avoiding flux-balance

- $\left\langle \frac{\mathrm{d}K}{\mathrm{d}\tau} \right\rangle$ computable directly (1$^{\text{st}}$ order)
- Looks like a flux-balance law:

$$\left\langle \frac{\mathrm{d}K}{\mathrm{d}\tau} \right\rangle = (\text{term at } \infty) + (\text{term at horizon})$$

- Carter constant for gravitational waves?
- Might make 2$^{\text{nd}}$ order calculations easier



[Mino, 2003], [Sago et al., 2006]

“Carter current” for scalar fields

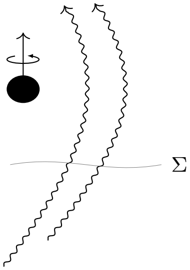
- $K = K_{ab}p^a p^b$
- $\mathcal{D} = \nabla_a(K^{ab}\nabla_b)$ satisfies $[\mathcal{D}, \Box] = 0$
- Conserved current:

$$j_K^a \equiv (\overline{\mathcal{D}\Phi})\nabla^a\Phi - \Phi\nabla^a\overline{\mathcal{D}\Phi}$$

- “Geometric optics”: $\Phi \propto e^{-i\vartheta/\epsilon}$, $\epsilon \to 0$
  $\implies$ scalar quanta propagating on rays

$$\int j_K^a \mathrm{d}\Sigma_a \propto \sum_{\substack{\text{scalar}\\\text{quanta}}} K$$

“Reduces to K for point particle”



[Carter, 1977]

Anatomy of the scalar Carter current

- Bilinear "Klein-Gordon current":

$$j^a(\Phi_1, \Phi_2) = \Phi_1 \nabla^a \overline{\Phi_2} - \overline{\Phi_2} \nabla^a \Phi_1$$

- Map on solution space: "symmetry operator"

$$\Box \mathcal{D} \Phi = \mathcal{D} \Box \Phi$$

  Φ solution $\implies$ $\mathcal{D}\Phi$ solution
- Conserved current: $j^a(\Phi, \mathcal{D}\Phi)$
- *All* of these steps can be generalized to linearized gravity!

New currents for linearized gravity

Bilinear current + symmetry operator:

$$\underbrace{{}_s\mathcal{C}j^a \equiv j^a\left({}_s\mathcal{C}\cdot\delta g, \overline{{}_s\mathcal{C}\cdot\delta g}\right)}_{\text{“Adjoint currents” } (s=\pm 2)}, \qquad \underbrace{{}_2\mathring{\mathcal{C}}j^a \equiv \sum_{s=\pm 2} j^a\left({}_s\mathring{\mathcal{C}}\cdot\delta g, \overline{{}_s\mathring{\mathcal{C}}\cdot\delta g}\right)}_{\text{“Projected adjoint current”}}$$

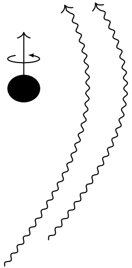
	Finite as $r\to\infty$		
Current	Outgoing waves	Ingoing waves	Local?
${}_2\mathcal{C}j^a$	$\times$	✓	✓
${}_{-2}\mathcal{C}j^a$	✓	$\times$	✓
${}_2\mathring{\mathcal{C}}j^a$	✓	✓	$\times$

Geometric optics

- "High-frequency" ansatz:

$$\delta g_{ab} = \mathrm{Re}\left\{ a\varpi_{ab}[1 + O(\epsilon)]e^{-i\vartheta/\epsilon} \right\} \quad (\epsilon \to 0)$$

- Gravitons propagating along rays
- ϖ_{ab} captures *polarization*; characterized by e_R, e_L with $|e_R|^2 + |e_L|^2 = 1$

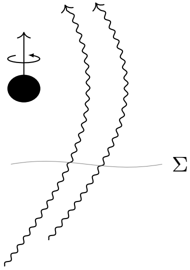


Geometric optics for our currents

- Integrating our currents yields

$$\int ...j^a \mathrm{d}\Sigma_a \propto \sum_{\text{gravitons}} K^4 \left(|e_R|^2 - |e_L|^2\right)$$

  (only valid for ${}_2\mathring{\mathcal{C}} j^a$ with caveats)
- $|e_R|^2 - |e_L|^2$ dependence: vanishes for linear polarizations
- Other currents in [G & Flanagan, 2020] similar



Bilinear currents: Symplectic product

- Lagrangian formulation for field ϕ
- Vary Lagrangian density:

$$\delta(\sqrt{-g}\,L) = \sqrt{-g}\,[\underbrace{E \cdot \delta\phi}_{\text{EoM}} + \overbrace{\nabla_a \theta^a(\delta\phi)}^{\text{“boundary term”}}]$$

- Vary θ^a:

$$\sqrt{-g}\,j^a(\delta_1\phi, \delta_2\phi) \equiv \delta_1[\sqrt{-g}\,\theta^a(\delta_2\phi)] - \delta_2[\sqrt{-g}\,\theta^a(\delta_1\phi)]$$

- $\delta_1\phi$, $\delta_2\phi$ satisfy linearized EoM $\implies$

$$\nabla_a j^a(\delta_1\phi, \delta_2\phi) = 0$$

[Burnett & Wald, 1990]

Bilinear currents: Symplectic product (examples)

- Generalizes Klein-Gordon current:

$$L = (\nabla\Phi)^2 \implies j^a(\delta_1\Phi, \delta_2\Phi) \sim \delta_1\Phi\nabla^a\delta_2\Phi - \delta_2\Phi\nabla^a\delta_1\Phi$$

- Bilinear current for metric perturbations:

$$L = \frac{1}{16\pi}R$$

$$\Downarrow$$

$$j^a(\underbrace{\delta_1 g, \delta_2 g}_{\text{metric pert.}}) = S^{abcdef}[\delta_1 g_{bc}\nabla_d\delta_2 g_{ef} - (1 \longleftrightarrow 2)]$$

where

$$S^{abcdef} = \text{(triple products of raised metric tensors)}$$

[Burnett & Wald, 1990]

Background: Black hole perturbation theory

- "Weyl scalars" $\delta\Psi_0$, $\delta\Psi_4$: components of curvature
- Characterize radiation:

$$\delta\Psi_4 \sim \underbrace{e^{i\omega(t-r)}/r}_{\text{outgoing waves}}, \qquad \delta\Psi_0 \sim \underbrace{e^{i\omega(t+r)}/r}_{\text{ingoing waves}}$$

- Rescaled versions have "decoupled" EoM:

$$_s\Omega \equiv \begin{cases} \delta\Psi_0 & s = -2 \\ (r - ia\cos\theta)^4 \delta\Psi_4 & s = 2 \end{cases}$$

$$\Downarrow$$

$$_s\Box\, {}_s\Omega = 8\pi\, {}_s\tau_{ab}\delta T^{ab}$$

(Teukolsky equation)

[Teukolsky, 1973]

Background: Black hole perturbation theory

Teukolsky equation: ${}_s\Box\, {}_s\Omega = 8\pi\, {}_s\tau_{ab}\delta T^{ab}$

- Two decoupled, complex *scalar* PDEs, easier than *tensor* PDE:
$$\mathcal{E}^{abcd}\delta g_{cd} = 8\pi\delta T^{ab}$$
- ${}_s\Box$ *separates* in r & θ:
$${}_s\Box = {}_s\mathcal{R}(r,t,\phi) + {}_s\mathcal{S}(\theta,t,\phi)$$
- Consider mapping ${}_sM : \delta g \mapsto {}_s\Omega$; then
$${}_s\Box\, {}_sM = {}_s\tau \cdot \mathcal{E}$$

[Teukolsky, 1973]

Adjoint symmetry operators

- Take adjoint:

$$ {}_s\tau \cdot \mathcal{E} = {}_s\Box \, {}_sM \quad \Longrightarrow \quad \mathcal{E}^\dagger \cdot {}_s\tau^\dagger = {}_sM^\dagger \, {}_s\Box^\dagger $$

- Key insight: $\mathcal{E}^\dagger = \mathcal{E}$, ${}_s\Box^\dagger = {}_{-s}\Box$
- Multiply on right by ${}_{-s}M$:

$$ \mathcal{E} \cdot \underbrace{{}_s\tau^\dagger \, {}_{-s}M}_{\equiv {}_s\mathcal{C}} = {}_sM^\dagger \, {}_{-s}\Box \, {}_{-s}M = {}_sM^\dagger \, {}_{-s}\tau \cdot \mathcal{E} $$

- δg vacuum solution $\Longrightarrow$ ${}_s\mathcal{C} \cdot \delta g$ vacuum solution!

[Wald, 1978]

Asymptotics & projection operators

- ${}_s\mathcal{C} \cdot \delta g$ not always well-behaved:
  - Outgoing waves: ${}_2\mathcal{C} \cdot \delta g \sim r$
  - Ingoing waves: ${}_{-2}\mathcal{C} \cdot \delta g \sim r$
- Projection operators:

$$\mathcal{P}^{\text{down}}(\text{outgoing waves}) = 0,$$
$$\mathcal{P}^{\text{up}}(\text{ingoing waves}) = 0$$

- Non-local: care about $r \to \infty$ behavior!
- New symmetry operators:

$${}_2\mathring{\mathcal{C}} \equiv {}_2\tau^\dagger \, \mathcal{P}^{\text{down}} \, {}_{-2}M, \quad {}_{-2}\mathring{\mathcal{C}} \equiv {}_{-2}\tau^\dagger \, \mathcal{P}^{\text{up}} \, {}_2M$$

New currents for linearized gravity

Bilinear current + symmetry operator:

$$\underbrace{{}_{s\mathcal{C}}j^a \equiv j^a\left({}_s\mathcal{C}\cdot\delta g,\ \overline{{}_s\mathcal{C}\cdot\delta g}\right)}_{\text{“Adjoint currents” } (s=\pm 2)}, \qquad \underbrace{{}_{2\mathring{\mathcal{C}}}j^a \equiv \sum_{s=\pm 2} j^a\left({}_s\mathring{\mathcal{C}}\cdot\delta g,\ \overline{{}_s\mathring{\mathcal{C}}\cdot\delta g}\right)}_{\text{“Projected adjoint current”}}$$

Current	Finite as $r\to\infty$: Outgoing	Ingoing	Local?	Geometric optics
${}_{2\mathcal{C}}j^a$	×	✓	✓	$K^4(\lvert e_R\rvert^2 - \lvert e_L\rvert^2)$
${}_{-2\mathcal{C}}j^a$	✓	×	✓	
${}_{2\mathring{\mathcal{C}}}j^a$	✓	✓	×	

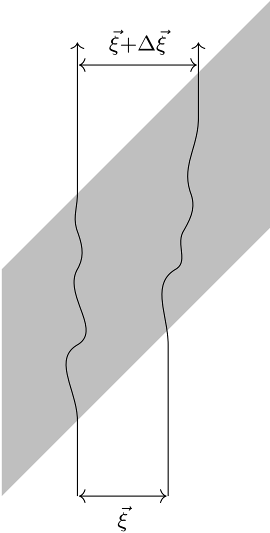
Conclusions and outlook for part I

- Found collection of conserved currents for linearized gravity associated with Carter constant (generalized notion of L^2)
- Geometric optics result *suggestive* of relation to point particle Carter constant
- Future work: how do these currents behave when coupled to sources (e.g. point particles)?

Outline

The gravitational wave memory effect

- Change in separation due to gravitational waves [Zel'dovich & Polnarev, 1974]
- Observable by LIGO & pulsar timing arrays in future
- Two ways to think about it:
  - Change in metric: $\Delta\xi_i \sim \Delta[\delta g_{ij}]\xi^j$
  - Integrated curvature: $\Delta\xi^\mu \sim \iint R^\mu{}_{\alpha\nu\beta}u^\alpha u^\beta \xi^\nu$



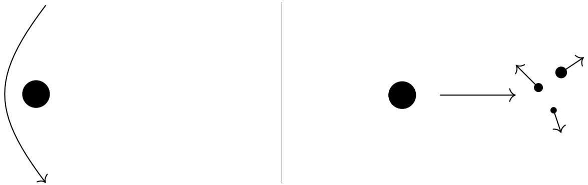
Physical sources: Linear memory

Quadrupole formula: $\delta g_{ij} = 2\ddot{Q}_{ij}/r + O(1/r^2)$

- What if $\ddot{Q}_{ij}$ differs before and after wave passes by?
- $Q_{ij} \sim m x_i x_j$, so

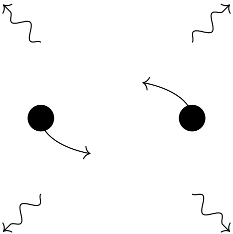
$$\Delta\ddot{Q}_{ij} \sim m\Delta\left[v_i v_j\right]$$

- v_i changes direction $\implies$ memory!
- *Unbound systems*, particles flying off to infinity



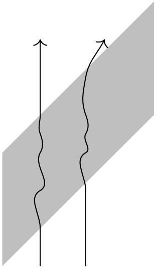
Physical sources: Nonlinear memory

- Memory still exists for *bound* systems; the "particles flying off to infinity" are *gravitational waves*
- Measurable by LIGO

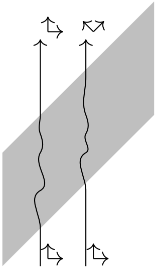


[Christodoulou, 1991], [Thorne, 1992]

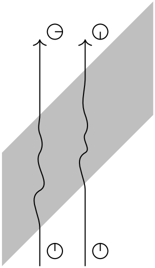
“New memory effects”



Relative boost
[Grishchuk & Polnarev, 1989]
$\sim \int R^{\mu}{}_{\alpha\nu\beta} u^{\alpha} u^{\beta} \xi^{\nu}$



Relative rotation
[Flanagan & Nichols, 2014]
$\sim \int R^{\mu}{}_{\alpha\nu\beta} u^{\alpha} \xi^{\beta}$



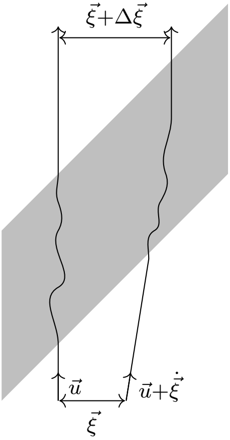
Proper time shift
[Strominger & Zhiboedov, 2014]
$\sim \int R_{\alpha\beta\gamma\delta} u^{\alpha} \xi^{\beta} u^{\gamma} \xi^{\delta}$

Subleading displacement memory effect

- Assume initial $\dot{\xi}^\mu \neq 0$:

$$\Delta\xi^\mu \sim \overbrace{\iint R^\mu{}_{\alpha\nu\beta} u^\alpha u^\beta \xi^\nu}^{\text{usual memory}} + (\tau' - \tau)\dot{\xi}^\mu + \underbrace{\iiint R^\mu{}_{\alpha\nu\beta} u^\alpha u^\beta \dot{\xi}^\nu}_{\text{“subleading” memory}}$$

- E & B decomposition on a sphere:
  CoM memory [Nichols, 2018] &
  Spin memory [Pasterski, et al.; 2016]



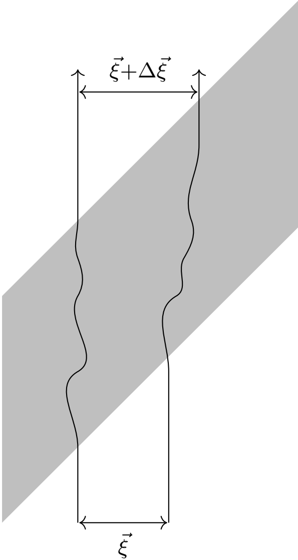
Classification of observables

Persistent observables

- Measurement over a time interval that vanishes in absence of radiation
- This talk: flat-to-flat transitions (e.g. plane waves)

Special case: Memory observables

- Associated with *boundary symmetries* (asymptotic, horizons, etc.)
- Example: nonzero when conjugate "conserved quantities" differ
- Not the focus of this talk



[Flanagan, G, Harte, Nichols; 2019]

Summary of persistent observables

Observable	Integrals of Riemann tensor	Scaling as $r \to \infty$	Associated with symmetry?
Displacement	2	$1/r$	Supertranslations
Relative boost	1	$1/r^2$	No
Relative rotation	1	$1/r^2$	No
Relative proper time	1	$1/r^2$	No
Subleading displacement	3	$1/r$	Superrotations
Curve deviation	1–3[a]	?	?
Angular momentum holonomy	1–3[a]	?	?
Spinning test particle	1–2	?	?

[a]With acceleration, the number of time integrals is 4 and higher.

Old observables

New observables

Summary of persistent observables

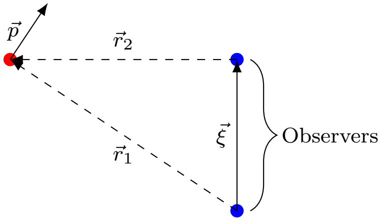
Observable	Integrals of Riemann tensor	Scaling as $r \to \infty$	Associated with symmetry?
Displacement	2	$1/r$	Supertranslations
Relative boost	1	$1/r^2$	No
Relative rotation	1	$1/r^2$	No
Relative proper time	1	$1/r^2$	No
Subleading displacement	3	$1/r$	Superrotations
Curve deviation	1–3[a]	?	?
Angular momentum holonomy	1–3[a]	?	?
Spinning test particle	1–2	?	?

[a]With acceleration, the number of time integrals is 4 and higher.

Focus of talk: how angular momentum encodes old observables

Angular momentum and displacement memory

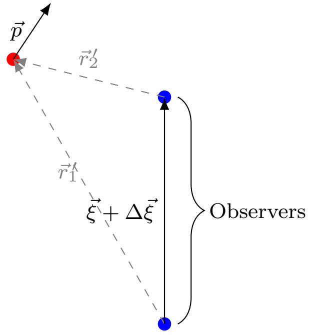
Before burst:



$$\Delta \vec{L} = \vec{\xi} \times \vec{p}$$

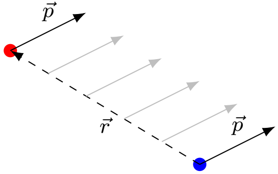
Angular momentum and displacement memory

After burst with memory:



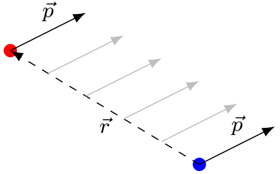
$$\Delta\vec{L}' - \Delta\vec{L} = \Delta\vec{\xi} \times \vec{p}$$

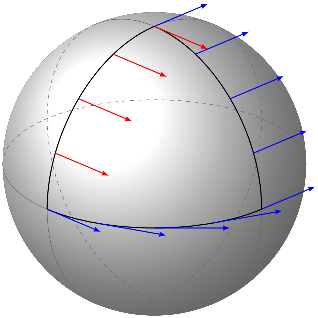
Linear momentum and parallel transport



- $\vec{p}$ is a vector at particle, not observer
- $\vec{r} \times \vec{p}$ requires *parallel transport*

Linear momentum and parallel transport





- $\vec{p}$ is a vector at particle, not observer
- $\vec{r} \times \vec{p}$ requires *parallel transport*

- Parallel transport is *path-dependent* in curved spacetimes: "holonomy"

Parallel transport and relative boost/rotation

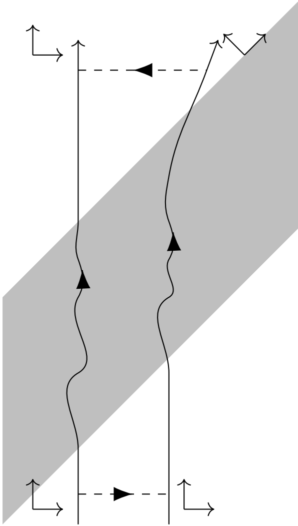
- Basis vectors: parallel-transported along all curves

$$k^b \nabla_b v^a = 0$$

- Geodesic motion: four-velocity parallel-transported:

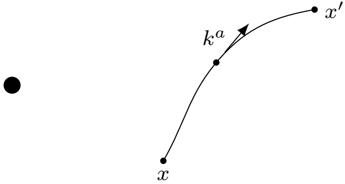
$$u^b \nabla_b u^a = 0$$

(comparisons *also* parallel transport)



Angular momentum transport

Values at x' from those at x:



$$\underbrace{k^b \nabla_b P^a = 0}_{\substack{\text{parallel transport} \\ \text{(boost/rotation)}}}, \qquad \underbrace{k^c \nabla_c J^{ab} = 2P^{[a}k^{b]}}_{\substack{\text{“}\vec{r}\times\vec{p}\text{” part} \\ \text{(displacement/proper time)}}}$$

Angular momentum holonomy

- Solving

$$k^b \nabla_b P^a = 0$$
$$k^c \nabla_c J^{ab} = 2P^{[a}k^{b]}$$

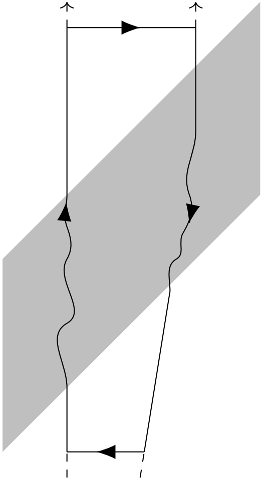
around loop gives "holonomy":

$$X^A \equiv \binom{P^a}{J^{ab}} \mapsto \overset{0}{\Lambda}{}^A{}_B X^B$$

- Encodes old observables:

$$\overset{0}{\Lambda}{}^A{}_C = \begin{pmatrix} \Lambda^a{}_c & 0 \\ 2\Delta\chi^{[a}\Lambda^{b]}{}_c & \Lambda^{[a}{}_{[c}\Lambda^{b]}{}_{d]} \end{pmatrix}$$

$\Lambda^a{}_b$: Boost & rotation
$\Delta\chi^a$: Displacement & proper time



Generalizing angular momentum transport

- Transport law inspired by flat spacetime
- Add in non-trivial curvature coupling:

$$k^b \nabla_b P^a = 0$$

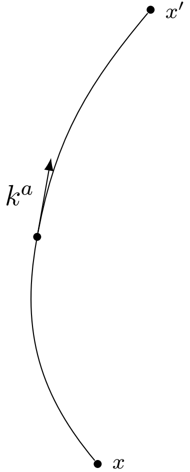
$$\Downarrow$$

$$k^b \nabla_b P^a = -\overset{\varkappa}{K}{}^a{}_{bcd} k^b J^{cd}$$

($\overset{\varkappa}{K}{}^a{}_{bcd}$ constructed from curvature)

- Two examples of $\overset{\varkappa}{K}{}^a{}_{bcd}$:

$$\underbrace{\tfrac{1}{2} R^a{}_{bcd}}_{\text{“dual Killing”}}, \qquad \underbrace{-\tfrac{1}{4} R^a{}_{bcd} + \tfrac{1}{2} \delta^a{}_{[c} R_{d]b}}_{\text{“path-independent”}}$$



Dual Killing transport

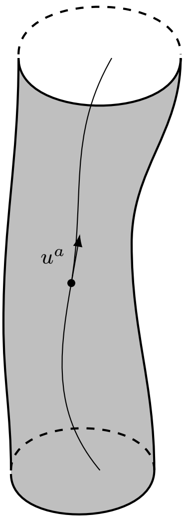
- Small body: P^a and J^{ab} arise as *multipoles*
- Evolution given by *Mathisson-Papapetrou equations*:

$$u^b \nabla_b P^a = -\frac{1}{2} R^a{}_{bcd} u^b J^{cd}$$
$$u^c \nabla_c J^{ab} = 2P^{[a} u^{b]}$$

(exactly dual Killing transport!)

- Connected with symmetries ("Killing vectors"):

$$x^\mu \to x^\mu + \epsilon \xi^\mu \text{ preserves metric}$$
$$\implies P^a \xi_a + \frac{1}{2} J^{ab} \nabla_a \xi_b = \text{const.}$$



Generalized angular momentum holonomy

- Solving

$$k^b \nabla_b P^a = -\check{K}^a{}_{bcd} k^b J^{cd}$$
$$k^c \nabla_c J^{ab} = 2P^{[a} k^{b]}$$

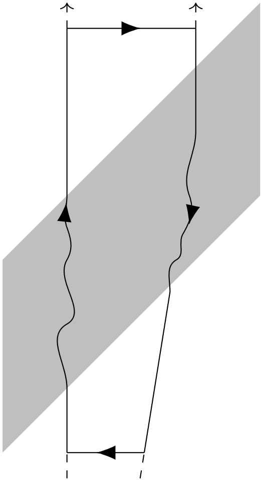
around loop gives holonomy:

$$X^A \mapsto \check{\Lambda}^A{}_B X^B$$

- Encodes old & new observables
- "Path-independent transport":

$$\check{K}^a{}_{bcd} = -\tfrac{1}{4} R^a{}_{bcd} + \tfrac{1}{2} \delta^a{}_{[c} R_{d]b}$$
$$\implies \check{\Lambda}^A{}_B \text{ can be trivial asymptotically}$$

(w/ no radiation)



Conclusions and outlook for part II

- Persistent observables: generalized enduring effects on gravitational wave detectors
- Origin-dependence of angular momentum encodes old (and introduces new) observables
- Future work: considering these and other observables near null infinity; relationship to asymptotic symmetries?

Thank you!